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Probabilistic Assessment of The Failure of Timber Bridges

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Abstract:

For a given loading, the ratio of the expected failure load of a laminated timber bridge, and the analytical failure load obtained by assuming the modulus of rupture (MOR) and modulus of elasticity (E_L) to have their respective means values, is called here the reduction factor. The value of this factor is always less than 1.0, and depends upon the transverse deflection profile of the bridge. A procedure is presented in this paper by which a realistic assessment of the reduction factor can be obtained by using test data for MOR and E_L for any species of wood.

Key Words: timber bridges, probabilistic assessment, failure loads, modulus of rupture, modulus of elasticity, reduction factor

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November 1985

PROBABILISTIC ASSESSMENT OF THE FAILURE OF TIMBER LAMINATED BRIDGES

LESLIE G. JAEGER[1] and BAIDAR BAKHT[2]

ABSTRACT

For a given loading, the ratio of the expected failure load of a laminated timber bridge, and the analytical failure load obtained by assuming the modulus of rupture (MOR) and modulus of elasticity (E_L) to have their respective means values, is called here the reduction factor. The value of this factor is always less than 1.0, and depends upon the transverse deflection profile of the bridge. A procedure is presented in this paper by which a realistic assessment of the reduction factor can be obtained by using test data for MOR and E_L for any species of wood.

INTRODUCTION

There are two properties of timber beams which have a marked effect on the distribution of live loads in timber laminated bridges and on the ultimate load carrying capacity of such bridges; these are the modulus of elasticity and the modulus of rupture.

The modulus of elasticity (E_L) and modulus of rupture (MOR) vary quite widely from one timber beam to another. Figure 1 shows test values of E_L and MOR on seventy Red Pine specimens. These values were originally given in an unpublished report of the Ontario Ministry of Transportation and Communications, and were used recently in Ref. (1). As will be seen from Figure 1, E_L and MOR are rather strongly correlated, but not perfectly so. If they were perfectly correlated all points in Figure 1 would appear on a single line. The variability in E_L and MOR, and more particularly the variability in E_L/MOR, has the effect of significantly reducing the ultimate load carrying capacity of the bridge,

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as compared to what it would be if all beams had the same modulus of elasticity and modulus of rupture, equal to their mean values for the species concerned. This significant reduction can be related to the amount of scatter that is shown in Figure 1. The reduction factor is also dependent upon the deflection profile of a cross-section of the bridge.

This paper deals only with the estimation of the "expected" load capacity; estimation of the variance of this capacity is dealt with elsewhere.

DERIVATION OF THE METHOD OF ANALYSIS

As a first step the values of E_L and MOR given in Figure 1 are replotted on axes of $E_L/(E_L)_{mean}$ and $MOR/(MOR)_{mean}$ as shown in Figure 2(a). Then for a given species of timber the scatter of points can be approximated as lying within an area bounded by $r=r_1$ and $r=r_2$ and by $\theta=\alpha$ and $\theta=(90^o-\alpha)$, where r is a radial distance from the origin and θ is an angle measured clockwise from the $E_L/(E_L)_{mean}$ axis, as is shown in Figure 2. For Red Pine the angle α is about 30^o. It will be shown later that the limiting values r_1 and r_2 do not need to be estimated.

The behaviour depicted in Figure 2(a) may be contrasted with that of Figure 2(b) which shows what happens if the values of E_L and MOR are assumed to be perfectly correlated, i.e. if E_L/MOR is assumed to be constant. It will be seen that the imperfect nature of the correlation which occurs with Red Pine results in a "fan" of points rather than a single line.

The scatter of points over the area of Figure 2(a) is represented by a probability density function $p(r,\theta)$ which satisfies the relationship

$$\int_{\alpha}^{\frac{\pi}{2}-\alpha} \int_{r_1}^{r_2} p(r,\theta) r dr d\theta = 1 \qquad (1)$$

For simplicity two further assumptions are made about the nature of the of points in Figure 2(a). These are that the probability distribution can be split up into "variation with θ " and "variation with r" and that the "variation with θ" is symmetrical about the line $\theta = 45^{\circ}$. Both of these assumptions are reasonable in light of the observed test values for different species of timber.

It is fortunate for the purposes of estimation of failure load that the analysis is not very sensitive to the precise form of the probability function $p(r,\theta)$ that is chosen. This point is returned to later. The general nature of the probability distribution $p(r,\theta)$ is shown in figure 3(a). Then we require

$$p(r,\theta) = f(r)\phi(\theta) \tag{2}$$

where:

$$\int_{r_1}^{r_2} f(r) r\,dr = 1 \tag{3}$$

and:

$$\int_{\alpha}^{\frac{\pi}{2} - \alpha} \phi(\theta)\, d\theta = 1 \tag{4}$$

Figure 3(b) shows the general nature of the probability function $\phi(\theta)$. This function must be zero outside the range of values α and $(90^{\circ}-\alpha)$ and be symmetrical about $\theta = 45^{\circ}$. A suitable form for the purpose of analysis is

$$\phi(\theta) = c(\sin 2\theta - \sin 2\alpha) \tag{5}$$

for:

$$\alpha < \theta < 90^{\circ} - \alpha$$

The constant c in equation (5) is readily found by using equation (4). In the case of Red Pine, taking $\alpha = 30^{\circ}$, one then obtains

$$\phi(\theta) = 21.488(\sin 2\theta - \frac{\sqrt{3}}{2}) \tag{5a}$$

Having derived the necessary properties of the scatter of pairs of values of E_L and MOR, an investigation is now undertaken of ultimate failure loads for various patterns of transverse deflection. For each pattern of deflection, the actual expected value of ultimate failure load is related to the deterministic failure load which is obtained by taking all beams to have the same E_L and MOR, being the mean value for the timber species concerned. As a preliminary it may be noted that the ultimate failure load, M_{ult}, of a beam is directly proportional to its MOR. Hence in Figure 2(a) the horizontral axis can be $M_{ult}/(M_{ult})_{mean}$ as well as $MOR/(MOR)_{mean}$.

The Case of Uniform Deflection

The first pattern of tranverse deflection that is examined is that of uniform deflection, shown in Figure 4. If all beams are taken to be identical, then as the uniform deflection increases the bending moments accepted by the beams increase until all beams simultaneously reach $(M_{ult})_{mean}$. Hence in the deterministic approach the total ultimate bending moment capacity of the bridge is simply:

$$M_T = N(M_{ult})_{mean} \qquad (6)$$

When the scatter of values of E_L and MOR is taken into account, a very different behaviour emerges. The beams do not fail simultaneously. Since all beams deflect equally the moment accepted by a beam is proportional to its own E_L and its ability to withstand moment is proportional to its own value of MOR, (or, alternatively its own value of M_{ult}). The first beams to fail are hence those with the lowest value of MOR/E_L. Figure 5 is a repeat of Figure 2(a) with the addition of a failure boundary at $\theta=\beta$. As the uniform deflection shown in Figure 4 increases a radial line, starting at the vertical axis, rotates clockwise in Figure 5. For a certain deflection all of those beams which are to the left of the boundary in Figure 5 have already failed, whilst

those to the right have not. Similarly to Ref. (2) we assume that when a beam's ultimate moment capacity has been reached the beam fails and its moment carry capacity falls to zero.

The total bending moment being accepted by those beams that have not yet failed is readily found. We consider the small element $r\delta r\delta\theta$ shown at the point S in Figure 5. The number of beams expected in this area is $Np(r,\theta)r\delta r\delta\theta$. These beams are not yet at their failure moments; they are in fact taking a fraction (NP/NS) of their failure moments, with the notation of Figure 5. This follows because all beams having the same value of E_L take the same moments as one another, until failure. Using the notation shown in Figure 5 we have

$$NP = r\cos\theta\tan\beta \tag{7}$$

The bending moment accepted by the beams in the shaded area $r\delta r\delta\theta$ is

$$\delta M = \{Np(r,\theta)r\delta r\delta\theta\}r\cos\theta\tan\beta(M_{ult})_{mean} \tag{8}$$

Using equations (2) (3) (4) and (6) we then obtain, on integrating equation (8):

$$M = M_T\tan\beta\int_{r_1}^{r_2} f(r)r^2dr\int_{\beta}^{\frac{\pi}{2}-\alpha} \cos\theta\phi(\theta)d\theta \tag{9a}$$

which is conveniently written

$$M = k(\beta)M_T \tag{9b}$$

where

$$k(\beta) = \tan\beta\int_{r_1}^{r_2} f(r)r^2dr \int_{\beta}^{\frac{\pi}{2}-\alpha} \cos\theta\phi(\theta)d\theta \tag{9c}$$

The calculation of $k(\beta)$ using equation 9(c) is straightforward and is greatly simplified by the fact that the function f(r) does not need to be found explicitly. In fact the value $\int_{r_1}^{r_2} f(r)r^2dr$ can be determined from knowing the position of the centre of gravity of the probability distribution. Details of the evaluation of equation 9(c) are given in ref. (3).

It is also shown in Ref. (3) that the expression for k(β) can be simplified, on eliminating the integral with respect to r, to

$$k(\beta) = \frac{2 \tan \beta \int_{\beta}^{(\pi/2 - \alpha)} \cos \theta \, \phi(\theta) \, d\theta}{\int_{\alpha}^{(\pi/2 - \alpha)} (\sin \theta + \cos \theta) \, \phi(\theta) \, d\theta} \qquad (9d)$$

k(β) is readily evaluated from 9(d) for any assumed $\phi(\theta)$.

Although the details of the mathematics may not be of much interest to engineers, the underlying physics of how the collapse builds up certainly are. Referring again to Figures 4 and 5, for small values of deflection the boundary $\theta=\beta$ does not go through the area defined by the scatter of points; as deflection continues to increase, however, and the radial line continues its motion, a stage is reached at which the radial line reaches the angular position α of Fig. 5. From that point on the most vulnerable beams (i.e. those with the smallest values of MOR/E_L) begin to fail. For a time, increasing values of deflection, and the corresponding increases in the angle β , are accompanied by further increases in the load carrying capacity, but the rate of increase decreases steadily. Eventually a value of β is reached for which k(β) is a maximum, after which it reduces. The maximum value of k(β) gives the failure state of the bridge, in accordance with Equation 6.

For Red Pine, the equation for k(β) is given by Ref. (3) as

$$k(\beta) = 30.588 \tan\beta \left(\frac{2}{3} \cos^3\beta + \frac{\sqrt{3}}{2} \sin\beta - \frac{5}{6}\right) \qquad (10)$$

for values of β greater than 30°.

Figure 6 is a graph of ultimate load factor k(β) plotted against the angle β. The maximum is reached when β is 0.62 radians, i.e., 35.52°, at which point k(β) has the value 0.639. Hence for Red Pine in uniform deflection

$$M = 0.639 \, M_T = N \{0.639 (M_{ult})_{mean}\} \qquad (11)$$

From Equation (11), the ultimate failure load of the bridge is reduced, because of the scatter of values, to 0.639 of what it would have been if all beams had had the same M_{ult}, i.e. if all had had $(M_{ult})_{mean}$, and the same value of E_L, i.e., $(E_L)_{mean}$.

It should be emphasized that the reduction in the ultimate failure load as estimated by Equation (11) is not brought about by having beams which are, on the average, weaker than usual. The beams are taken to have an average strength which is entirely normal, and it is the scatter of (MOR/E_L) above and below the mean which gives rise to the progressive failure of the bridge and the reduction in its ultimate load-carrying capacity as compared with the "no scatter" situation.

The reduction factor of 0.639 which appears in Equation (11) is derived from the probability distribution of the quantity (MOR/E_L) and the ultimate failure moment M given by Equation (11) is in fact the mean value (i.e., the "expectation") of a random variable. The derivation of a failure load for design purposes requires the determination also of the standard deviation of this random variable. That determination is given in Ref. 3. It is sufficient to note here that for timber laminated bridges the Number, N, of beams is large and hence the variability of the ultimate failure load about the mean value given by Equation (11) is small. It is very instructive to repeat the analysis given above for other assumed probability distributions. Fortunately the reduction factor is found to be not very sensitive to such changes; for example if the function $\phi(\theta)$ is assumed to be constant for values of θ between α and $(90^o - \alpha)$ as shown in Figure 3(c), even this quite marked change only has the effect of lowering the reduction factor to about 0.577 when $\alpha = 30^o$.

Figures 3(b), 3(c) and 3(d) bring out clearly one important point. This is that the shift from the "perfect correlation" case of Figure 3(d) to either Figure 3(b) or Figure 3(c) gives a change in reduction factor from 1.00 to

0.639 or 0.577, which is very much larger than the difference between the latter two. This confirms that a fairly good estimate of the reduction factor can be obtained by taking a reasonable representation of the scatter with respect to θ .

The Case of Linearly Varying Deflection

The adoption of a completely uniform deflection pattern across a transverse cross-section is, of course, a highly idealized case, which can be regarded as one extreme. For purposes of comparison, the second deflection pattern considered is that of Figure 7, in which the deflection is assumed to vary in a straight line manner from zero at one side of the bridge to a maximum value at the other. If all beams are taken to be identical, with modulus of elasticity $(E_L)_{mean}$ and ultimate moment $(M_{ult})_{mean}$ then failure is reached when the deflection of the right hand end reaches the value which gives $(M_{ult})_{mean}$ at that end.

Making again the assumption that the moment of resistance of a failed beam falls immediately to zero, it is readily seen that attainment of $(M_{ult})_{mean}$ at the right hand end leads at once to collapse of the bridge and that

$$M_T = 0.5N\ (M_{ult})_{mean} \qquad (12)$$

In Equation (12) the factor 0.5 arises from the linear variation of deflection across the cross-section.

The estimation of total failure moment for this deflection pattern, taking into account the scatter in values of (MOR/E_L) follows the same general line as that for the uniform deflection cases, except that now each element of the cross-section has its own deflection and hence its own radial line on the plot of Figure 2. Figures 8(a) and 8(b) show the situation in which the deflection of the right hand end gives the radial

line $\theta=\beta$. Then a width $B(\tan\alpha/\tan\beta)$ of the cross-section is known to be still unfailed, whilst the right hand portion of the cross-section is composed of elements which have increasing probabilities of failure as one moves from left to right. Details of the estimation of the ultimate failure moment are once again given in Ref. (3); in the case of Red Pine the result is

$$M = 0.415\ N\ (M_{ult})_{mean} \qquad (13)$$

In this case it is seen from Equations (12) and (13) that the reduction factor in ultimate load carrying capacity is 0.830, as compared to 0.639 in the case of a uniform deflection. This relatively better performance is entirely to be expected since with the pattern of deflection now assumed, a fairly large fraction of the cross-section is known to be not involved in the initiation of failure, so that the influence of the scatter of values of (MOR/E_L) is much reduced.

The procedure for estimating ultimate moment capacity for any general deflection pattern, such as that shown in Figure 9, follows the same steps. The maximum deflection of the cross-section corresponds to a radial line $\theta=\beta$ in the plot of Figure 5. Any other point on the cross-section has a radial line $\theta=\lambda$ where $(\tan\lambda/\tan\beta)$ is the ratio of deflections for the two points concerned. Estimation of the ultimate moment capacity then follows using the method given in Reference 3.

In practice, the reduction factor for a uniform deflection, which is 0.639 in the case of Red Pine, is found to be the lowest reduction factor that occurs.

CONCLUSIONS

The failure loads of timber laminated bridges cannot be safely estimated by treating all of the beams as having the same (average) modulus of elasticity and same (average) modulus of rupture. For Red Pine a safe estimate of failure load is about 60% of the failure load obtained in that

way. The estimate of failure load is not very sensitive to the probability distribution that is assumed; provided that this distribution is fairly near the truth, the reduction factor due to the progressive nature of collapse can be predicted with reasonable accuracy.

It is noted that this paper deals with estimation of the "expections" of failure load i.e. with estimation of the mean value of a random variable. The estimation of the standard derivation of this random variable is outlined elsewhere (Ref. 3).

AKCNOWLEDGEMENTS

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(b) The research has also been supported significantly by the Ontario-Ministry of Transportation and Communications, Downsview, Ontario,as part of its on-going program of bridge research. This support is also gratefully acknowledged.

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3. Jaeger, L. G. and Bakht, B. Ultimate Failure Loads of Timber Bridges, Structural Research Report SRR-85-04, Ministry of Transportation and Communications, Downsview, Ontario, Canada, September 1985.

NOTATION

E_L Modulus of elasticity of a timber beam in the longitudinal direction

$(E_L)_{mean}$ Mean value of E_L for the species of timber concerned

MOR Modulus of rupture of a timber beam

$(MOR)_{mean}$ Mean value of MOR for the species of timber concerned

$E_L/(E_L)_{mean}$, $MOR/(MOR)_{mean}$ Coordinate axes for representation of probability distribution of a given species of timber

(r,θ) Polar coordinates in the plane of $E_L/(E_L)_{mean}$ and $MOR/(MOR)_{mean}$, with θ being measured from the $E_L/(E_L)_{mean}$

$p(r, \theta)$ A probability distribution

α Minimum value of θ for which the probability function $p(r, \theta)$ has non-zero values

$f(r)$, $\phi(\theta)$ Variations of $p(r, \theta)$ in the r and θ directions respectively

β The value of θ corresponding to the maximum deflection in the cross-section of a bridge; i.e., the maximum value of λ.

λ The value of θ corresponding to the deflection of a representative point on the cross-section of a bridge.

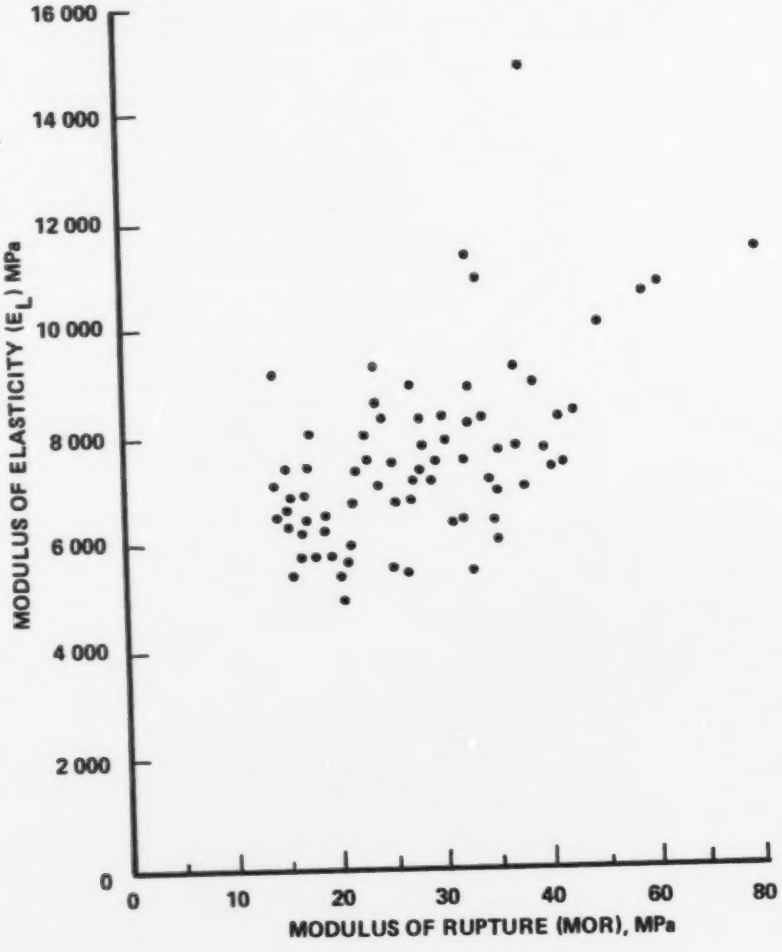
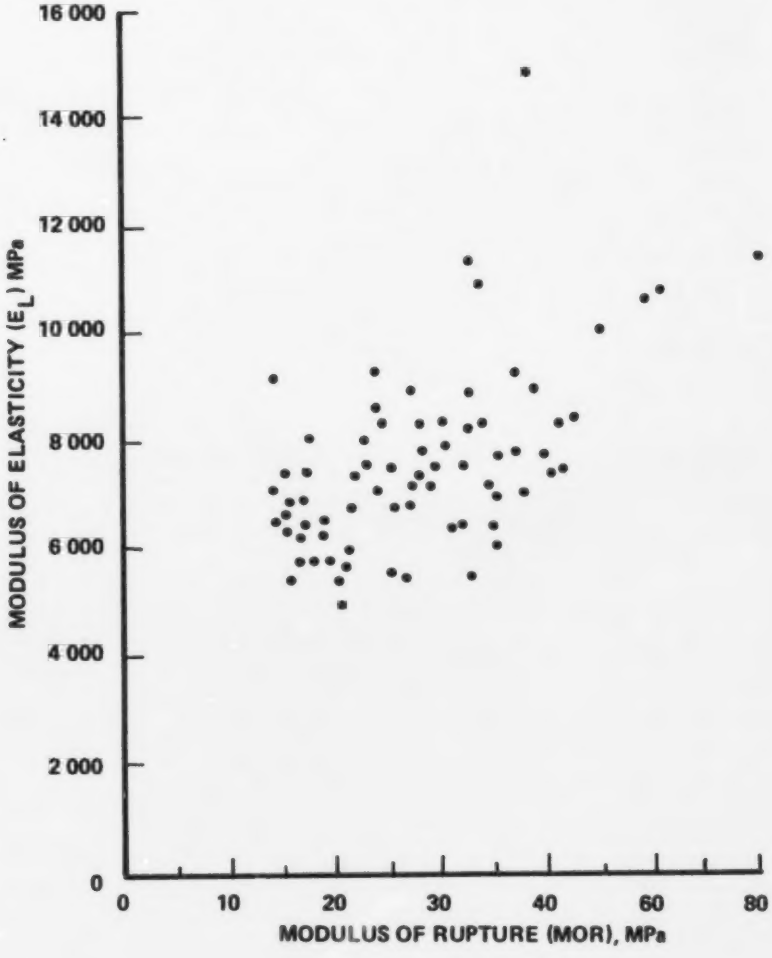


Figure 1/ Test Data on 70 Red Pine Specimens



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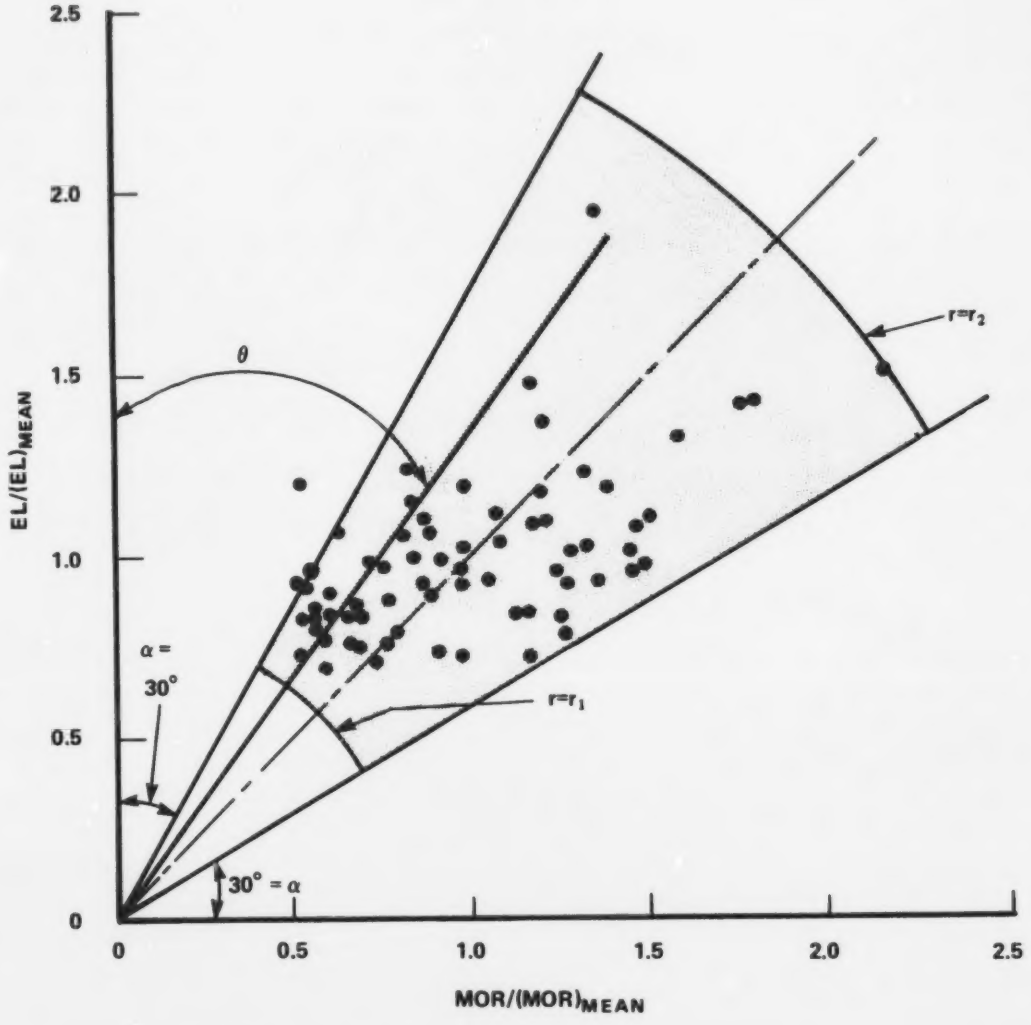


Figure 2(a)/ Data on Red Pine Wood

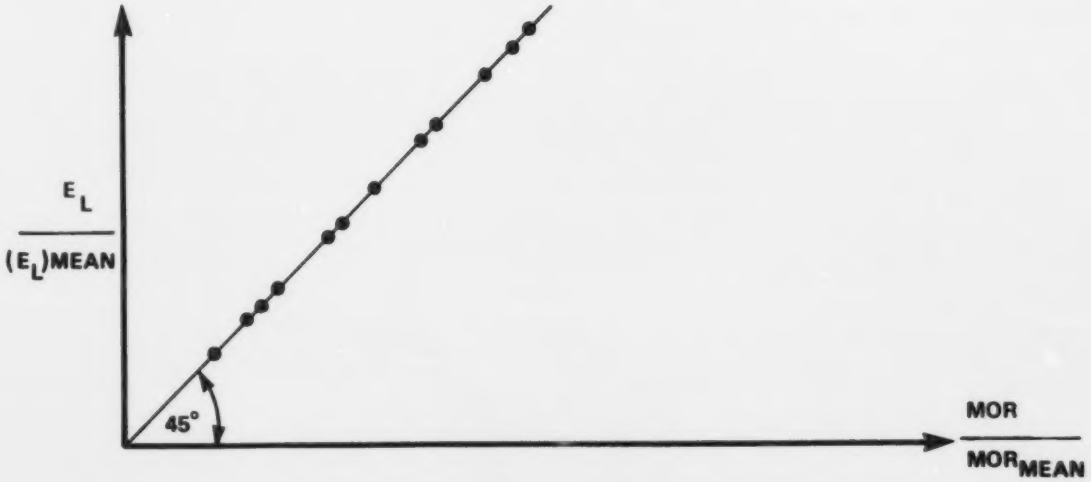


Figure 2(b)/ Perfect Correlation of E_L and MOR

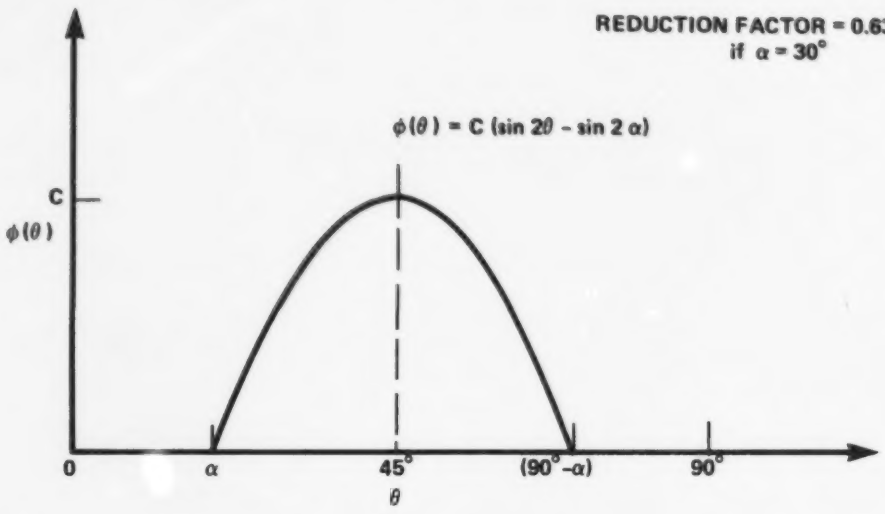


Figure 3(b)/ Plot of $\phi(\theta)$ Against θ

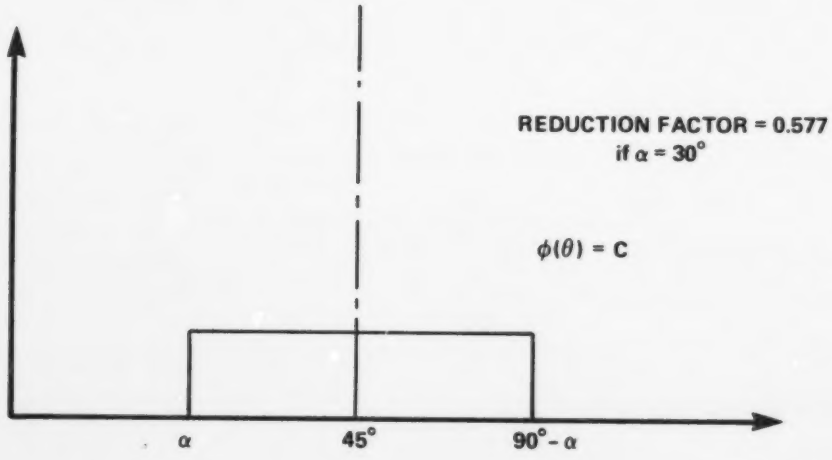


Figure 3(c)/ An Extreme Assumption for $\phi(\theta)$

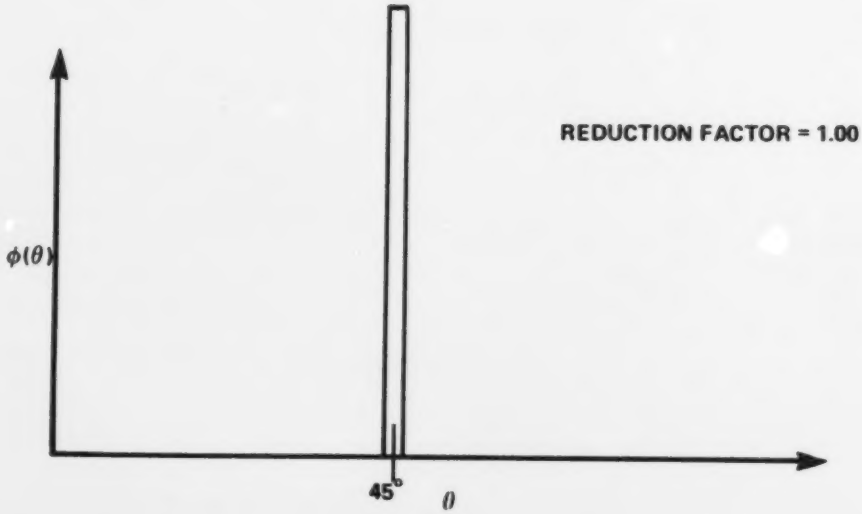


Figure 3(d)/ The Plot of $\phi(\theta)$ for Perfect Correlation of E_L and MOR
an Indefinitely High Spike of Vanishing width (Dirac Delta Function)

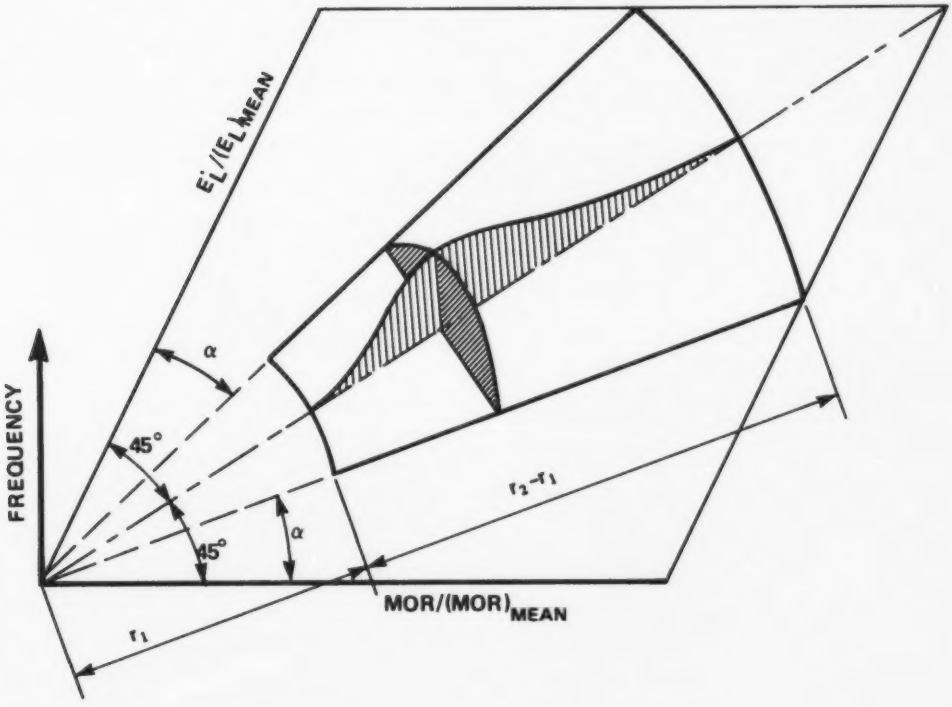


Figure 3(a)/ The Probability Function $p(r,\theta)$

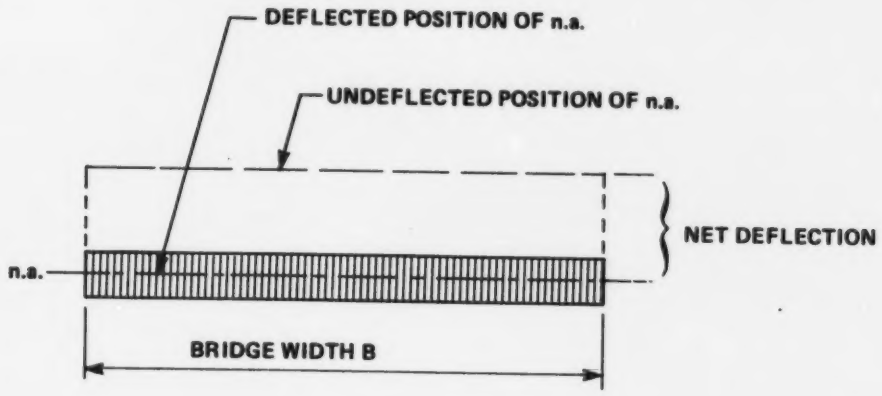


Figure 4/ Uniform Deflection Across a Bridge Cross-section

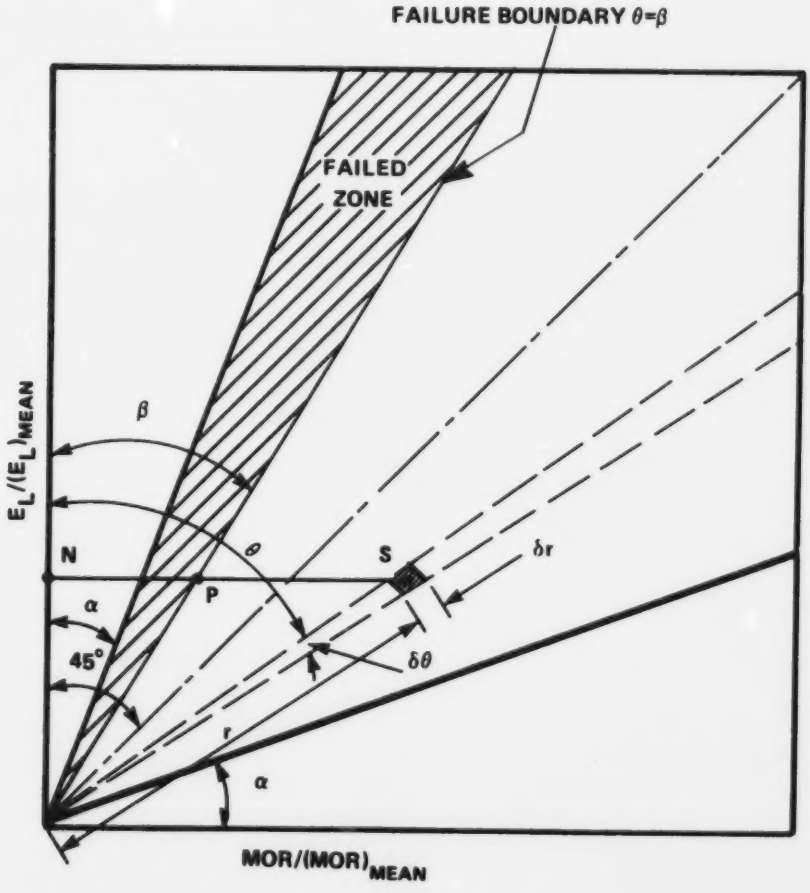


Figure 5 / Failed Zone

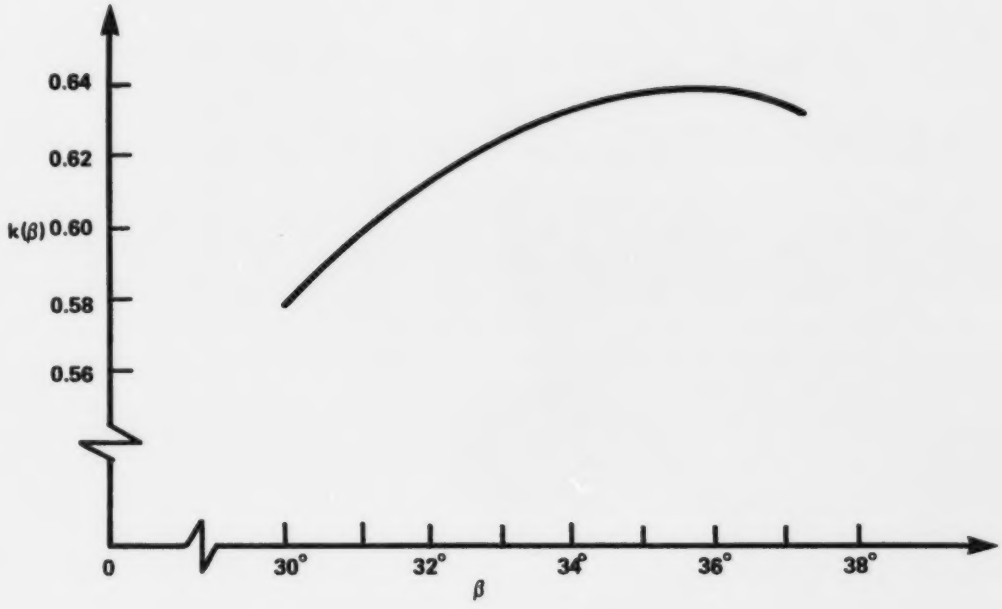


Figure 6/ Plot of k (β) Against β

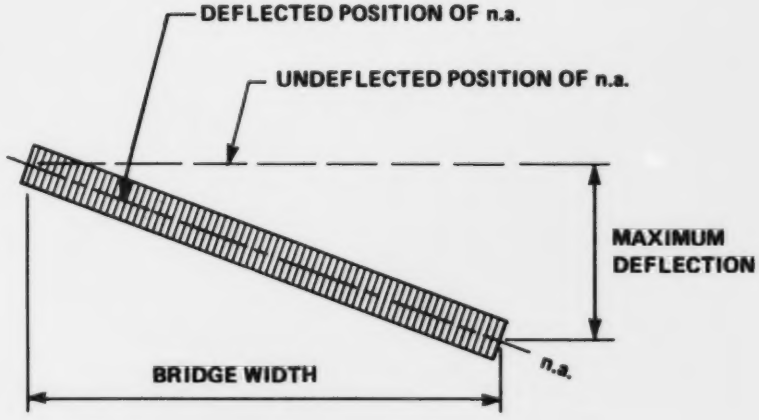
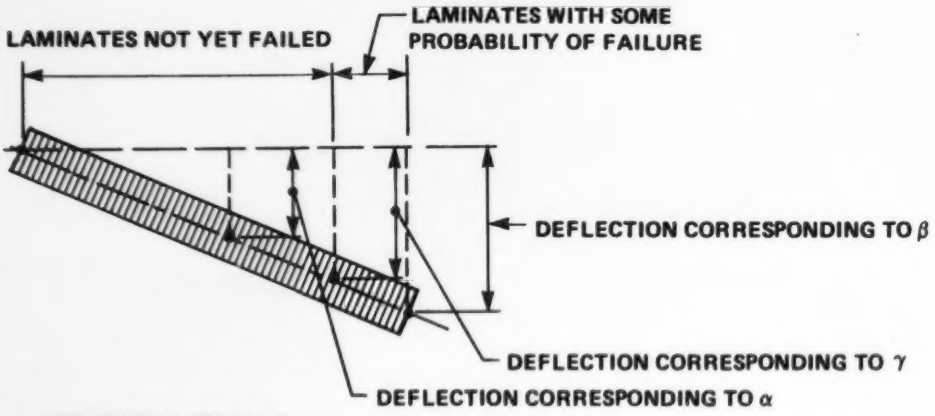
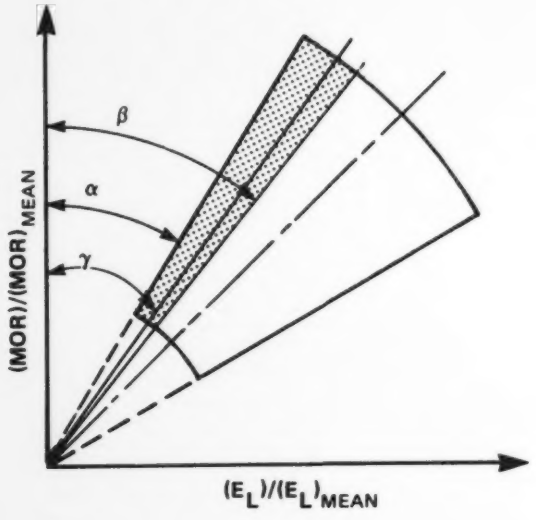


Figure 7/ Linear Deflection Across a Bridge Cross-section


LAMINATES NOT YET FAILED
LAMINATES WITH SOME PROBABILITY OF FAILURE
DEFLECTION CORRESPONDING TO β
DEFLECTION CORRESPONDING TO γ
DEFLECTION CORRESPONDING TO α

a. DEFLECTION PROFILE



b.

Figure 8/ Identification of Failure Zone

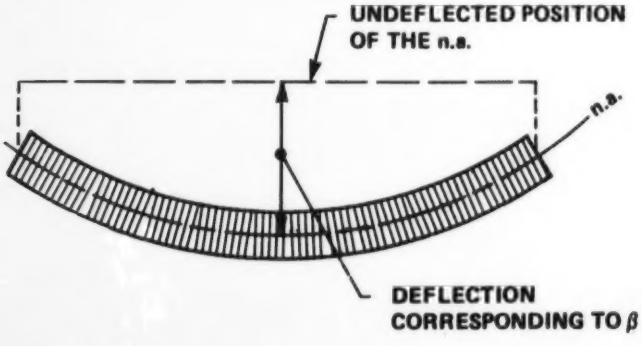


Figure 9/ A Non-uniform Deflection Profile